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THE HUREWICZ THEOREM

by A. V. Zarelua

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By A. V. Zarelua

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A. V. Zarelua

ABSTRACT

The author presents the Hurewicz theorem for weakly paracompact spaces, with proofs of the theorems and auxiliary propositions in the first part of his work. The second part deals with completely paracompact spaces and decompositions, again with proofs of the theorems and lemmas. Part 3 is concerned with the proofs of the Hurewicz theorem for producing paracompact decompositions.

Introduction

The work consists of three parts. In the first part the summation theorem is extended to point-finite open coverings,[1] while the Hurewicz theorem on the decrease in dimension during transformations is proven in a form given by Morita (ref. 1), but with the assumption of weak paracompactness[2] of the image (Morita's image is paracompact). In the third part the same theorem of Hurewicz is proven in a different form, established by Yu. M. Smirnov (ref. 3) (Theorem 6), for a certain wide class of transformations. Furthermore, as a corollary we obtain the same theorem for a sufficiently wide class of normal spaces which contain all strongly paracompact spaces[3] (the case considered by Nagami in reference 10), in particular for finally-compact (Lindelöf) spaces (ref. 3). From this we obtain an important inequality dim X ≤ and X[4] for the space X of the considered class (for strongly paracompact spaces this /17*

*Numbers given in margin indicate pagination in original foreign text.

[1]The covering is called point-finite, if each point is contained only in a finite number of elements of this covering.

[2]The space is called weakly paracompact, if each open covering admits an open point-finite covering.

[3]The covering is called star-finite, if each of its elements intersects only with a finite number of other elements of the same covering. The space is called strongly paracompact, if each of its open coverings admits a star-finite open covering.

[4]The dimension dim in this case is a dimension determined by means of finite open coverings, while the dimension ind (small) is an inductive dimension.

was established by Morita (ref. 2), for finally compact spaces by Yu. M. Smirnov (ref. 3) and for bicompact spaces by P. S. Alexandroff (ref. 4)).

The second part introduces and investigates the concept of sufficiently paracompact space and sufficiently paracompact decomposition (for such spaces and decomposition the Hurewicz theorem is proved in Part 3). For metrizable spaces this class of spaces produces strongly metrizable spaces, i.e., spaces containing a base which decomposes into a sum of countable number of star-finite open coverings,[1] apparently introduced by Morita. Several examples are presented, in particular an example of a strongly metrizable space which does not decompose into a sum of countable number of closed strongly paracompact subspaces.

Part 1. The Hurewicz Theorem for Weakly Paracompact Spaces

THEOREM 1. If the normal space R has a point-finite open covering ω such that $\dim \bar{U} \leqslant n$ for each $U \in \omega$, then $\dim R \le n$. /18

Before carrying out the proof we note that for closed point-finite coverings the assertion of theorem 1 is incorrect, because any T_1 space has a point-finite covering consisting of single point sets. The weakest limitation known to us, for which the assertion of theorem 1 is true for a closed covering, is that if F_λ is an arbitrary closed subset of the element $A_\lambda \in \alpha$, then the various sums of sets F_λ are closed in R (the property of "strong conservatism" is discussed in reference 6). The property of the local finite value of a covering is stronger than the property of conservatism (ref. 5, lemma 1) and stronger than the property of point finiteness considered here.

In order to prove the theorem we shall require the following two lemmas:

Lemma 1. Let ω be a point-finite open covering; then for each n = 1, 2, 3,... the set T_n, consisting of all those points, each of which belongs to not more than n elements of the covering ω, is closed.

Lemma 2. Let us assume that with the same assumptions U is the neighborhood of a set T_{k-1}; then the difference $T_k \setminus U$ is the body[2] of a closed discrete[3] system inscribed into the covering ω.

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[1]Regular spaces containing such a base are metrizable due to the Nagata-Smirnov theorem (ref. 5).
[2]The body of the system is a set of all spatial points belonging at least to one element of this system.
[3]The system of sets is called discrete, if each point in space has a neighborhood which intersects not more than one element of the given system.

Proof of the Lemma. The first lemma is obvious. We designate by U_λ various intersections over k different elements of the covering ω. It is clear that the intersection of sets U_λ is contained completely in $R\setminus T_{k-1}$. Therefore the sets $U_\lambda \cap T_k \setminus U$ are open in $T_k\setminus U$ and form the covering α of multiplicity factor 1 for the difference $T_k\setminus U$. Consequently each set $U_\lambda \cap T_k\setminus U$ is closed in $T_k\setminus U$, and consequently in R. From this we can see that α is a system which is discrete in $T_k\setminus U$, and consequently in R. The lemma has been proven.

Proof of the Theorem. Since $T_0 = \phi$, we can prove by induction the inequality $\dim T_k \le n$ for all k. From this, in view of the summation theorem, we find that $\dim R \le n$.

Thus, let us assume that $\dim T_{k-1} \le n$. Let us also assume that the closed set Φ lies in $T_k\setminus T_{k-1}$. In view of lemma 2 the set Φ is a body of the discrete closed system α inscribed into ω. Consequently each element of this system α has a dimension $\le$n and therefore $\dim \Phi \le n$ also. Since $\dim T_{k-1} \le n$, it follows from the Dowker theorem (ref. 7, (2.1)) that $\dim T_k \le n$ also. The theorem is proven.

Corollary. If space R is normal and is weakly paracompact, then loc dim R = dim R.[1] /19

Proof. It is clear that loc dim R $\le$ dim R. Let us assume that n = loc dim R. Then each point $x \in R$ has a neighborhood Ox such that $\dim [Ox] \le n$. We inscribe a point-finite covering $\omega = \{U_\lambda\}$, into the covering $\{Ox\}$ and into the former we inscribe the combined closed point-finite covering $\alpha = \{A_\lambda\} : A_\lambda \subseteq U_\lambda$ according to the Leffchetz lemma (ref. 8, p. 44). By taking neighborhoods OA_λ of type F_σ such that $OA_\lambda \subseteq U_\lambda$ we obtain a point-finite covering $\omega' = \{OA_\lambda\}$, for which $\dim OA_\lambda \le n$ for all values of λ. This means that $\dim R \le n$. The corollary has been proven.

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[1]The local dimension loc dim R of the space R is defined by Dowker as the smallest of the numbers n possessing a property whereby at each point $x \in R$ there is a neighborhood Ox, such that $\dim [Ox] \le n$ (ref. 7).

As in the work of Dowker (ref. 7), we find that the equality loc dim R = dim R is also true in the following cases: (1) if the normal space is a sum of a countable number of closed weakly paracompact sets; (2) if the normal space is the sum of a locally finite system of weakly paracompact sets, of which all except one are closed.

We call the dimension of the transformation g: $X \to Y$ the upper boundary of dimensions $\dim g^{-1}(y)$ of the complete prototypes of all points $y \in Y$. It is natural to designate this dimension by dim g.

THEOREM 2. Let us assume that g is a closed[1] transformation of a normal space X to a weakly paracompact normal space Y; then $\dim X \leq \dim g + \operatorname{ind} Y$.[2]

Lemma 3. Let f be the transformation of a closed set Φ of normal space X into a sphere S^n, while $\{\Gamma_k\}$ is the countable open covering of space X, such that the boundaries $\operatorname{Fr}\Gamma_k$ of its elements have the dimension $\dim \operatorname{Fr}\Gamma_k \leq n - 1$ and that transformation f may be extended to each of the sets $\Phi \bigcup [\Gamma_k]$; then transformation f may be extended over the entire space X.

The proof of the lemma contained in reference 3 consists of transposing the well-known Hurewicz proof to the present case (ref. 9, p. 125), while observing the necessary precautions.

Proof of the Theorem. The theorem is obvious for the case when Ind Y = -1. Let us assume that it is true for all cases when $\operatorname{Ind} Y \leq m - 1$, and let us prove it for the case when Ind Y = m. Let Φ be a closed set of space X, and let f be the transformation of this set to the sphere S^n, where $n = m + \dim g$. According to the theorem of P. S. Alexandroff (ref. 3, p. 163) we must show that f can be extended over the entire space X. Since $\dim g^{-1}(y) \leq n$, for each point $y \in Y$, the transformation f may be extended to the set $\Phi \bigcup g^{-1}(y)$ and consequently to some neighborhood U_y of the set $\Phi \bigcup g^{-1}(y)$. Because the transformation g is closed, the sets $Oy = Y \setminus g(X \setminus U_y)$ constitute the open /20 covering of space Y. Since Y is weakly paracompact we can inscribe the point-finite covering ω into this covering. Let us assume that T_k are sets determined in lemma 1 while $S_k = g^{-1}(T_k)$. Now we shall show by induction that for any k = 0, 1, 2,... open sets Γ_k exist which satisfy the following five conditions:

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[1]Continuous transformation is called closed if the image of any closed set is also a closed set.

[2]By Ind Y we represent the "great" inductive dimension of space R obtained by induction over the closed sets.

(1) $\bigcup_{k<l} S_k \subseteq \bigcup_{k<l} \Gamma_k$ for all l,

(2) $\dim \operatorname{Fr}\Gamma_k \leq n - 1$,

(3) the image Γ_k, $G_k = g(\Gamma_k)$ is open in Y,

(4) $g^{-1}(G_k) = \Gamma_k$,

(5) the transformation f can be extended to $\Phi \cup [\Gamma_k]$.

When $k = 0$, we have $S_0 = \phi$. Therefore we can let $\Gamma_0 = \phi$.

Let us assume that for all $i < k$ the sets Γ_i which satisfy conditions 1-5 have been constructed. Let us construct the set Γ_k. Let $F = S_k \setminus \bigcup_{i<k} \Gamma_i$. In view of property 4 we have $F = g^{-1}(T_k \setminus \bigcup_{i<k} G_i)$. According to lemma 2 the set $T_k \setminus \bigcup_{i<k} G_i$ is a body of the closed discrete system $\{A_\lambda\}$, inscribed into the covering ω. Therefore the closed discrete system $\{B_\lambda\}$ where $B_\lambda = g^{-1}(A_\lambda)$ is inscribed into the covering $\{U_\gamma\}$. Thus set F is a body of system $\{B_\lambda\}$. Since transformation f is extendable to $\Phi \cup B_\lambda$ for any λ, in the discreteness system it is extended to $\Phi \cup F$ and to some neighborhood H of set $\Phi \cup F$. The set $W = Y \setminus g(X \setminus H)$ is the neighborhood of the image $g(F)$, since the transformation g is closed. This means that there is a set G_k which is open in Y such that $g(F) \subseteq G_k \subseteq [G_k] \subseteq W$ and $\operatorname{Ind} \operatorname{Fr}G_k \leq m - 1$. For the prototype G_k, $\Gamma_k = g^{-1}(G_k)$ we have $F \subseteq \Gamma_k \subseteq [\Gamma_k] \subseteq H$ and $g(\operatorname{Fr}\Gamma_k) \subseteq \operatorname{Fr} G_k$. Since the closed subsets of weakly paracompact spaces are weakly paracompact, $\dim \operatorname{Fr} \leq \dim g + m - 1 = n - 1$ due to the first inductive proposition. Thus the constructed sets $\Gamma_0, \ldots, \Gamma_k$ satisfy conditions 1-5. Continuing this construction we obtain a countable covering $\{\Gamma_k\}$ of space X, which satisfies the conditions of lemma 3. Consequently the transformation f may be extended over the entire space X and consequently $\dim X \leq n$. The theorem has been proven.

From this we can obtain the following corollary. (See remarks pertaining to the corollary of theorem 1.)

Corollary. If g is a closed transformation of normal space X into space Y, then $\dim X \leq \dim g + \operatorname{Ind} Y$ in the following two cases: (1) space Y is the sum of a countable number of closed weakly paracompact sets, (2) space Y is a body of local-finite system of weakly paracompact sets, of which all except one are closed.

Part 2. Completely Paracompact Spaces and Breakdowns

Let us say that the system of sets β is weakly inscribed into the covering α, if we can select a subsystem from it which is a covering inscribed into /21 α. Thus, for example, any base of any space is weakly inscribed into any of its open coverings. We call a regular space completely paracompact, if we can inscribe a system which breaks down into a sum of a countable number of star-finite coverings into any of its open coverings.

This propositon is justified by the following results.

THEOREM 3. Any regular completely paracompact space is paracompact; any strongly paracompact space is completely paracompact.

Proof. The second proposition is obvious, because it is assumed that any open covering of a strongly paracompact space may contain the star-finite covering. To prove the first proposition in the light of the well-known theorem of Michael (ref. 11), it is sufficient to prove the following lemma.

Lemma 4. Into any open covering of a completely paracompact space we can inscribe an open covering which decomposes into the sum of a countable number of discrete subsystems.

We inscribe weakly into an arbitrarily open covering γ of completely paracompact space, a system ω which decomposes into a sum of a countable number of star-finite open coverings ω_i. It is easy to see (ref. 12) that each covering ω_i decomposes into the sum of countable or finite subsystems $\omega_{i\lambda}$, whose bodies do not intersect in pairs and consequently are open-closed sets. The elements of each such system are designated by natural numbers. Then each element U of the system ω will have three subscripts: i, λ and k, where the first refers to the number of covering ω_i to which it belongs, the second designates the subsystem $\omega_{i\lambda}$, and the third designates the number of the element U inside the system $\omega_{i\lambda}$. It is obvious that any system of elements with fixed subscripts i and k is discrete. Therefore all coverings ω decompose into the sum of a countable number of discrete subsystems. Consequently the subcovering of covering ω inscribed into γ also decomposes into a sum of a countable number of discrete subsystems. Lemma 4 and theorem 3 are proven.

We call a regular space strongly metrizable, if it contains the base of open sets which decompose into a sum of countable number of star-finite coverings. Since any star-finite open covering is locally finite, in view of the Nagata-Smirnov theorem (ref. 5) any strongly metrizable space is metrizable.

Lemma 5. A metrizable space is strongly metrizable when, and only when, it is completely paracompact.

Proof. Let ω be the base of strongly metrizable space which decomposes to a sum of countable number of star-finite open coverings. It is clear that it is weakly inscribed into any open covering of the given space. Conversely let γ_i be a sequence of open ϵ_i coverings of metric space where $\epsilon_i \to 0$. Let ω_i be weakly inscribed into γ_i. Obviously the association $\bigcup \omega_i$ is a base and decomposes into the sum of a countable number of star-finite coverings. The lemma has been proven.

It is easy to see that the product of the countable number of strongly metrizable spaces is strongly metrizable, and that each subspace of any /22 strongly metrizable space is strongly metrizable. Nagata has shown (ref. 13) that for a strongly paracompact metric space these properties are not satisfied: the product of the interval (open) (0.1) and the Baire space B^τ of noncountable weight τ is not strongly paracompact. At the same time the space is an open subset of the product of the segment (0.1) and the same space B^τ, which, as we can easily verify, is strongly paracompact. As shown initially by Morita (ref. 14, theorem 2.3), for all strongly metrizable spaces with weight $\leq\tau$ there is a universal space which is the product of the Hilbert cube I^∞ and the Baire space B^τ of weight τ. It is even strongly paracompact, like the product of the bicompact and strongly paracompact space (ref. 15).

Example 1. Strongly metrizable space which is not the sum of a countable number of closed strongly paracompact sets.

Let $\prod I_i$ be the topological product of the countable number of (open) intervals I_i. It is metrizable and has a countable base. The unknown space S is the product of the product $\prod I_i$ and the Baire space B^τ of the uncountable weight τ. Since it is the product of two strongly metrizable spaces, it is also strongly metrizable. Every base neighborhood V of space $\prod I_i$ is the product of a finite number $i < k$ of intervals I_i' and the product $\prod\limits_{i\geqslant k} I_i$. To obtain the base neighborhood W of space S we must multiply the neighborhood V in its obtained form by the base neighborhood $B = O(\xi, 1/m)$ of the Baire space B^τ. Since every base neighborhood B of the Baire space is closed and homeomorphous to the entire Baire space B^τ, we can, by taking in each interval I_i', $i < k$ and in each interval I_i, $i > k$, obtain from the point p_i a subset $\prod\limits_{i\neq k} p_i \times I_k \times B$ of neighborhood W, which is closed in S and homeomorphous to space $I \times B^\tau$, and consequently not strongly paracompact. Thus in any neighborhood of space S there is a subset which is closed in S and not strongly paracompact. It follows from this that any closed strongly paracompact subset of space S is nowhere dense in S. However, S is complete. Consequently it cannot be the sum of a countable number of closed strongly paracompact sets, which is what we had to prove.

For completeness we shall prove the following theorem.

THEOREM 4. Any subset of type F_σ of a completely paracompact space is sufficiently paracompact.

Proof. Let A_1 be the closed sets of sufficiently paracompact space R and let $B = \bigcup A_i$. Let γ be a covering of set B open in B and let Γ be sets open in R, such that $\Gamma \cap B \in \gamma$. Let, finally, the covering γ_i for each i consist of all sets Γ which intersect with A_1 and of sets $R \setminus A_i$. Into each covering γ_i we can weakly inscribe the covering ω_i, which decomposes into a sum of countable /23 number of star-finite coverings. Let us designate by $B \cap \omega_i$ the covering of set B consisting of all sets $U \cap B$ where $U \in \omega_i$. It is clear that the association $\omega = \bigcup B \cap \omega_i$ decomposes into a sum of countable number of star-finite coverings of set B. Let us prove that ω is weakly inscribed into γ. Indeed, let ω_i' be the subcovering of covering ω_i inscribed into γ_i. We designate by ω' the combination of all such intersections $U \cap B$, for which with certain i $U \in \omega_i'$ and $U \cap A_i \neq \phi$. It is easy to see that ω' is the covering of the set B and that $\omega' \subseteq \omega$. The covering ω' is inscribed into γ, because when $U \in \omega_i'$ and $U \cap A_i \neq \phi$, the set U is contained in one of the sets Γ. The theorem has been proven.

We note that contrary to the property of paracompactness the property of complete paracompactness (as well as the property of strong paracompactness) is not preserved during the closed transformation to either side, as shown by the following example.

Example 2. The sum of a noncountable number of segments of the same length emerging from one point O, in which distances are measured only along the segments ("metrizable hedgehog"), is a metrically coupled space. Due to the coupling, any star-finite open covering of this "hedgehog" is not more than countable. Therefore, if the "metrizable hedgehog" were sufficiently paracompact, it would also be strongly metrizable. Then it would have a countable base, which it actually does not. By transforming the entire "metrizable hedgehog" into one point, we obtained a closed transformation into a sufficiently paracompact space. The second space "nonmetrizable hedgehog" we obtain in exactly the same way by taking a noncountable number of segments of the same length emerging from a single point O. The arbitrary neighborhood of point O is the sum of half intervals emerging from the point O along each segment. The neighborhoods of remaining points are conventional neighborhoods of half intervals, obtained by ejecting point O from all segments.

The constructed space, "non-metrizable hedgehog," as we can easily verify, will be normal (even inheritably normal), but not a metrizable space, because at point 0 the first axiom of countability is not satisfied. The "non-metrizable hedgehog" is also coupled and consequently is not sufficiently paracompact. By taking a discrete noncountable set of segments we obtain a strongly paracompact and consequently a completely paracompact space. By marking one end on each of these segments and attaching all selected ends to one point we obtain a closed transformation of a strongly paracompact space into a "non-metrizable hedgehog."

We now proceed with further necessary derivations.

The decomposition of space is the combination of closed sets which do not intersect in pairs and whose sum is equal to the entire space. Each continuous transformation g of space X into space Y produces the decomposition of this space consisting of complete prototypes $g^{-1}(y)$ of all points $y \in Y$. We call the covering γ of space R the decomposition φ, if this decomposition φ is inscribed into γ. Finally we call the decompositon φ of space R completely paracompact, if /24 into any open decomposition covering ω we can weakly inscribe an open covering of space R, which decomposes into the sum of a countable number of star-finite space coverings R. The importance of this concept is that for closed transformations producing sufficiently paracompact decomposition the Hurewicz theorem is valid (Part 3). To prove this basic theorem and to derive the necessary corollaries, the following propositions are required.

Lemma 6. Let us assume that we have a closed transformation g of space X into space Y; if one of these spaces is sufficiently paracompact, the decomposition which produces the transformation of g is also sufficiently paracompact.

Proof. In the case of complete parcompactness of space X the decomposition produced by transformation g is completely paracompact, even if we do not consider the proposition that it is closed. Now let us assume that space Y is completely paracompact and that $\gamma = \{\Gamma\}$ is the covering of the decomposition $\varphi = \{g^{-1}(y)\}$, of the produced transformation g. For each point $y \in Y$ we select some one set Γ_y containing the prototype $g^{-1}(y)$. Because transformation g is closed, set $Oy = Y \setminus g(X \setminus \Gamma_y)$ is the neighborhood of point y. Into the covering $\{Oy\}$ obtained for space Y we can weakly inscribe an open covering ω, which decomposes into the sum of a countable number of star-finite coverings. It is easy to see that the complete prototypes of the covering elements ω will produce the unknown covering of space X, which is inscribed weakly into γ. The lemma has been proven.

Example 3. The closed transformation of an incompletely paracompact space into an incompletely paracompact space producing a completely paracompact breakdown.

In the "metrizable hedgehog" of example 2 we take any one segment AO and transform the points which lie on all remaining segments of the "hedgehog" into point O and retain the points of the segment AO in their place. We obtain a closed transformation of the "metrizable hedgehog" into segment AO. For space X we take the sum of noncountable numbers of discretely arranged "metrizable hedgehogs." In each of these we isolate one segment $A_\lambda O_\lambda$. We transform, by this method, each of the "metrizable hedgehogs" into the designated segment $A_\lambda O_\lambda$ and then join all points O_λ into one point O. It is clear that we shall obtain for the image Y a "non-metrizable hedgehog" of example 2, and that neither space X nor space Y are completely paracompact. The final transformation as the superposition of two closed transformations will also be a closed transformation. It is easy to verify that the decomposition produced by this transformation is completely paracompact. With some complication it is possible to achieve a space Y which is metrizable.

Part 3. The Hurewicz Theorem for Producing Paracompact Decompositions

The following theorem is of an auxiliary nature.

THEOREM 5. Let us assume that in a normal space for each neighborhood O of each element F of decomposition φ there is a neighborhood V, such that $F \subseteq V \subseteq O$ and $\dim \mathrm{Fr}V \leq n - 1$, where FrV is the boundary of neighborhood /25
V; then, if decomposition φ is completely paracompact and if $\dim F \leq n$ for each element $F \in \varphi$, $\dim R \leq n$.

To carry out the proof we need the following.

Lemma 7. If into an open covering η we can weakly inscribe an open covering which decomposes into a sum of countable number of star-finite coverings, then into η we can inscribe an open covering such that it decomposes into the sum of a countable number of discrete systems in such a way that the boundary of each of the elements lies on the boundary of some covering element η.

Proof of the Lemma. Let us assume that covering ω, which decomposes into a sum of a countable number of star-finite coverings ω_i, is weakly inscribed into the open covering η. As in the case of lemma 4 we number the elements U of the covering ω by means of three subscripts i, λ and k. The sums $H_{i\lambda} = \bigcup_k U_{i\lambda k}$ are open-closed sets. For each element $U_{i\lambda k}$ inscribed into η subcoverings ω', $\omega' \subseteq \omega$, we place one corresponding element $V \in \eta$ such that $U_{i\lambda k} \subseteq V$. We designate it by $V_{i\lambda k}$. Let $\Gamma_{i\lambda k} = H_{i\lambda} \cap V_{i\lambda k}$. Obviously $\mathrm{Fr}\,\Gamma_{i\lambda k} \subseteq \mathrm{Fr}\,U_{i\lambda k}$, and the sets $\Gamma_{i\lambda k}$ form a covering of space R inscribed into η. It is easy to see that the system of sets $\Gamma_{i\lambda k}$ with fixed subscripts i and k is discrete, which was to be proven.

Proof of the Theorem. Let φ be a completely paracompact decomposition of the normal space R satisfying all propositions of the theorem. Let f be the transformation of an arbitrarily chosen, closed set Φ into the sphere S_m. Since

dim F $\leq$ n for each element F of decomposition φ, transformation f may be extended to the closed set $\Phi \cup F$, and consequently to its neighborhood O. In view of our conditions there exists (for each $F \in \varphi$) a neighborhood V such that $F \subseteq V \subseteq [V] \subseteq O$ and dim FrV $\leq$ n - 1. Sets V comprise a covering η of decomposition φ. Therefore, due to the complete paracompactness of decomposition φ and due to lemma 7, we can inscribe into the covering η a covering which decomposes into the sum of a countable number of discrete systems γ_i, so that dim FrΓ

$\leq$ n - 1 for each element $\Gamma \in \bigcup \gamma_i$. For each discrete system γ_i we produce a set

$\Gamma_i = \bigcup_{\Gamma \in \gamma_i} \Gamma$. It is clear that $[\Gamma_i] = \bigcup_{\Gamma \in \gamma_i} [\Gamma]$ and that $\mathrm{Fr}\,\Gamma_i = \bigcup_{\Gamma \in \gamma_i} \mathrm{Fr}\,\Gamma$. Consequently

$\dim \mathrm{Fr}\,\Gamma_i \leqslant n - 1$. We continue the transformation f for each set $\Phi \cup [\Gamma]$, and consequently for the sets $\Phi \cup [\Gamma_i]$ for any i. In view of lemma 3 the transformation

f is extended over the entire space R. Consequently dim R $\leq$ n, which was to be proven.

THEOREM 6. Let g be a closed transformation of normal space X into normal space Y; if transformation g produces a completely paracompact breakdown of space X, dim X $\leq$ dim g + ind Y.

The proof will require the following.

Lemma 8. Let us assume that $\varphi = \{F\}$ is the decomposition of space X and φ' is its subsystem, such that the sum $A = \bigcup_{F \in \varphi'} F$ is closed; then the com- /26

plete paracompactness of decomposition φ produces a complete paracompactness of decomposition φ' on A.

Proof. Let γ' be the covering of decomposition φ' which is open in A. For each set $\Gamma' \in \gamma'$ let Γ be a set open in X, such that $\Gamma \cap A = \Gamma'$. Let the system γ consist of an addition $X \setminus A$ and of all sets Γ. The system γ is the covering of the decomposition φ, and into it we can weakly inscribe the covering of ω space X, which decomposes into the sum of a countable number of star-finite coverings. Then the system ω', consisting of all intersections $U \cap A$, where $U \in \omega$ is the covering of set A, weakly inscribed into γ' and decomposing into the sum of a countable number of star-finite coverings of set A. This proves that decomposition φ' is completely paracompact. The lemma has been proven.

Proof of the Theorem. If ind Y = -1, the theorem is correct. Let us assume that it is true for all cases when ind Y < k, and let us prove it in the case when ind Y = k. Let O be an arbitrary neighborhood of an arbitrarily selected prototype $g^{-1}(y)$. Because the transformation g is closed, there is a

neighborhood Oy of the point y, such that $g^{-1}(Oy) \subseteq O$ and such that ind FrOy $\leq k - 1$. According to the lemma, the decomposition produced by the transformation g on the complete prototype $g^{-1}(FrOy)$ is completely paracompact, which together with the induction proposition gives us an inequality dim $g^{-1}(FrOy) \leq m + k - 1$, where m = dim g. Consequently, this also gives us an inequality

dim $Frg^{-1}(Oy) \leq m + k - 1$. Applying theorem 5 we find that dim $X \leq m + k$. The theorem has been proven.

Corollary 1. If g is a closed transformation of normal space X into space Y, the inequality dim X ≤ dim g + ind Y takes place in the following two cases: (1) space X or space Y is the sum of the countable number of closed completely paracompact sets; (2) space X or space Y is the sum of completely paracompact sets constituting the local finite system, of which all except perhaps one are closed.

This corollary is proved by means of lemma 6, using the same approach as that of Dowker, which we have already mentioned.

Corollary 2. For any normal space X which satisfies one of the conditions of the preceding corollary, and particularly for completely paracompact space X, the inequality dim X ≤ ind X is valid.

From this and from the well-known Katetov-Morita theorem (ref. 16) we obtain one other corollary.

Corollary 3. For any metrizable space X, which decomposes into the sum of countable number of completely paracompact (i.e., strongly metrizable) closed sets, the equality dim X = ind X = Ind X is valid.

Remark. For metrizable spaces the second condition of the first corollary is reduced to the first condition.

The following theorem is closely associated with the problems considered.

THEOREM 7. If for a completely paracompact space the sum theorem is valid for the dimension Ind, then Ind X = ind X.

The proof is based on the following lemma. /27

Lemma 9. Let X be a completely paracompact space with dimension ind X ≤ n; then for any closed set F and any of its neighborhood U there is a neighborhood V, such that $F \subseteq V \subseteq U$ and such that the boundary FrV decomposes into a sum of a countable number of closed sets C_i with dimension ind $C_i \leq n - 1$.

Proof. In our propositions we can inscribe into the binary covering $\{R \setminus F, U\}$ according to lemma 7, a covering α which decomposes into the sum of a countable number of discrete systems α_i, such that ind FrG ≤ n - 1, if $G \in \alpha$. Into the covering α we can also inscribe covering β which also decomposes into the sum of a countable number of discrete systems β_i, such that ind FrH ≤ n - 1, if $H \in \beta$, while the closure of each element $H \in \beta$ lies in some element

$G \in \alpha$. We designate by γ_{ij} the combination of all those elements of systems β_i, each of which together with the closure lies in one of the elements of system α_i. We also let $\omega_{ij} = \alpha_i$.

Having established a mutual single-valued relationship between the set of binary indices ij and the set of all natural numbers k, we shall designate the systems γ_{ij} and ω_{ij} by γ_k and ω_k, respectively. Furthermore we let G_k be the sum of all those sets of systems ω_k which have a non-empty intersection with F, and we let $\Gamma_k = \bigcup_{H \in \gamma_k} H$. Because the system considered is discrete, we find that $[\Gamma_k] = \bigcup_{H \in \gamma_k} [H]$, $\mathrm{Fr}\,\Gamma_k = \bigcup_{H \in \gamma_k} \mathrm{Fr}\,H$ & $\mathrm{ind}\,\mathrm{Fr}\,\Gamma_k \leqslant n-1$. In the same way we find that $F \cap [\Gamma_k] \subseteq G_k$ and $\mathrm{ind}\,\mathrm{Fr}\,G_k \leqslant n-1$. Now we let $V_1 = G_1$, $V_k = G_k \setminus \bigcup_{i<k} [\Gamma_k]$ for $k > 1$. We shall show finally that the sum $V = \bigcup V_k$ is the desired neighborhood. It is clear that $V \subseteq U$, because if $F \cap G \neq \phi$, then $G \subseteq U$ for each element $G \in \bigcup \omega_k$. It is clear that $F \subseteq \bigcup G_k$. Let $x \in F \cap (G_k \setminus \bigcup_{i<k} G_i)$. Then $x \in V_k$, since $F \cap \bigcup_{i<k} [\Gamma_i] \subseteq \bigcup_{i<k} G_i$. This means that $F \subseteq V$. From the construction set Γ_i intersects only the finite number of sets V_k and $X = \bigcup \Gamma_i$. Therefore the system of sets V_k is locally finite. This means that $\mathrm{Fr}\,V \subseteq \bigcup \mathrm{Fr}\,V_k$. However, $\mathrm{Fr}\,V_k \subseteq \mathrm{Fr}\,G_k \bigcup_{i<k} \mathrm{Fr}\,\Gamma_i$. Consequently, $\mathrm{Fr}\,V \subseteq \bigcup \mathrm{Fr}\,G_k \cup \bigcup \mathrm{Fr}\,\Gamma_k$, which was to be proven.

Theorem 7 follows from this lemma by simple induction.

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REFERENCES

1. Morita, K. Closed Mappings and Dimension. Proc. Japan Acad., Vol. 32, No. 3, 161-165, 1956.

2. --- Dimension of Normal Spaces. II, J. Math. Soc. Japan, Vol. 2, No. 1-2, 16-33, 1950.

3. Smirnov, Yu. M. Certain Relationships in the Theory of Dimensions (Nekotoryye sootnosheniya v teorii razmernosti). Matem. Sb., Vol. 29 (71), 157-172, 1951.

4. Alexandroff, P. S. The Theorem of Addition in the Theory of Bicompact Spaces (Teorema spozheniya v teorii bikompaktnykh prostranstv). Soobshch. AN Gruz SSR, Vol. 2, 1-5, 1941.

5. Smirnov, Yu. M. Metrization of Topological Spaces (O metrizatsii topologicheskikh prostranstv). Uspekhi Matem. Nauk, Vol. VI, No. 6, (46), 100-111, 1951.

6. Morita, K. Spaces Having Weak Topology with Respect to Closed Covering. Proc. Japan Acad., Vol. 29, No. 10, 537-543, 1953.

7. Dowker, C. H. Local Dimension of Normal Spaces. Quart. J. Math., Vol. 6, No. 22, 101-120, 1955.

8. Leffchetz, S. Algebraic Topology (Algebraicheskaya topologiya). Moscow, I. L., 1949 (AMS Colloquium Publication No. XXVII, New York, 1942).

9. Hurewicz, W. and Wallman, H. Dimension Theory (Teoriya razmernosti). Moscow I. L., 1949 (Princeton, 1941).

10. Nagami, K. Some Theorems in Dimension Theory for Nonseparable Spaces. J. Math. Soc. Japan, Vol. 9, No. 1, 80-92, 1957.

11. Michael, E. A Note on Paracompact Spaces. Proc. AMS, Vol. 4, No. 3, 831-838, 1953.

12. Smirnov, Yu. M. Strongly Paracompact Spaces (O sil'no parakompaktnykh prostranstvakh). Izv. AN SSSR, seriya matem., Vol. 20, 253-274, 1956.

13. Nagata, J. Imbedding Theorem for Nonseparable Metric Spaces. J. Inst. Pol., Osaka City Univ., Vol. 8, No. 1, Series A, 9-14.

14. Shediva, V. Collectively Normal and Strongly Paracompact Spaces (O kollektivno-normal'nykh i sil'no parakompaktnykh prostranstvakh). Chekh. Matem. Zhurnal, Vol. 9, (84), 50-61, 1959.

15. Begle, E. G. A Note on S-spaces. Bull. AMS, Vol. 55, 577-579, 1949.

16. Katetov, M. Dimension of Metric Spaces (O razmernosti metricheskikh prostranstv). DAN SSSR, Vol. 79, No. 1, 189-191, 1951.

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